Existence of Equilibrium for Shared Goods

John A. Weymark

Department of Economics, Vanderbilt University,
VU Station B #351819, 2301 Vanderbilt Place,
Nashville, TN 37235-1819, USA.
E-mail: john.weymark@vanderbilt.edu

February 2025

Abstract. A shared good is an impure public good in which personalized consumptions are produced by groups using a sharing technology. Rivalry in consumption is captured by the shape of this technology. Private goods and pure public goods are special cases in which there is complete rivalry and no rivalry, respectively. A competitive shared goods equilibrium is defined in which there are markets for all goods, there are personalized prices for the consumption of shared goods, and both firms and groups are profit maximizers. When all shared goods are private (resp. public), this equilibrium is a Walrasian (resp. Lindahl) equilibrium. Sufficient conditions for the existence of a competitive shared goods equilibrium are identified. An alternative equilibrium concept in which groups behave cooperatively towards their beneficiaries is also considered and an equilibrium existence theorem for it is established.

Journal of Economic Literature classification numbers. C62; D51; H41

Keywords. impure public goods; shared goods; existence of equilibrium; Lindahl equilibrium; Walrasian equilibrium

1. Introduction

In a Walrasian equilibrium for private goods (Walras, 1954), consumers and firms take prices as given when making their demand and supply decisions, with prices adjusting to equilibrate the aggregate demand and supply for each good. In a Lindahl equilibrium for private and public goods (Lindahl, 1958), the markets for private goods operate in the same way. However, for the public goods, prices are personalized and they are adjusted so that each agent (consumer or firm) wants the same quantity.[1]

Many goods exhibit neither the perfect rivalry in consumption of private goods nor the complete nonrivalness of pure public goods—they are impure public goods. Examples include parks, roads, and concerts. Impure public goods take many forms. With a local public good (Tiebout, 1956), there is no rivalry in consumption within a community, but nobody benefits from the provision of the good outside of it. With a club good (Buchanan, 1965), there are congestion effects that depend on who consumes the good or the number of individuals consuming it.

In this article, an equilibrium concept is introduced for a class of impure public goods that have been considered by Buchanan (1966, 1968) and Weymark (2004), what the latter calls *shared goods*. Shared goods include as special cases both private goods and pure public goods. The new equilibrium concept—*a competitive shared goods equilibrium*—coincides with that of Walras when the goods that are shared are private and with that of Lindhal when they are public. The provision of shared goods is done by profit-maximizing groups who transform shared goods facilities into personalized consumptions of shared goods. Sufficient conditions for the existence of a competitive shared goods equilibrium are identified.

An alternative equilibrium concept—*a cooperative shared goods equilibrium*— in which the groups that provide shared goods behave cooperatively towards their beneficiaries is also considered. In this equilibrium, the groups that provide shared goods charge each consumer a lump-sum amount for the shared goods that it provides him with and chooses its sharing plan

[1] van den Nouweland et al. (2002) and van den Nouweland (2015) argue that Lindahl's equilibrium concept makes use of personalized cost shares for the production of public goods, not personalized prices for their consumption, and so corresponds to what is known as a ratio equilibrium. In keeping with the standard understanding of a Lindahl equilibrium, I regard it as employing personalized prices even if if that is not an accurate description of what Lindahl wrote.

and lump-sum charges so that no other choice on its part could result in a Pareto improvement. This equilibrium concept extends that of a public competitive equilibrium proposed by Foley (1967, 1970) for public goods so that it applies to shared goods. It is shown that a cooperative shared goods equilibrium exists whenever a competitive shared goods equilibrium exists.

Buchanan's proposal for modeling impure public goods is based on Marshall's theory of joint supply (Marshall, 1920). When there is joint supply, one good has associated with it several consumption goods. A well-known example is that of sheep, whose associated consumer goods are mutton and wool. Buchanan applied Marshall's analysis of joint supply to the provision of impure public goods. Buchanan regarded such a good as having associated with it person-specific consumptions. For example, a fire station provides fire protection services to nearby individuals, with the amount of protection provided to some particular individual depending on his proximity to the station. The distribution of these benefits depends on the location chosen for the station.

Buchanan's (1966; 1968) discussion and analysis of shared goods is somewhat informal. Weymark (2004) formally models Buchanan's proposal using the concept of a sharing technology, which specifies the distributions of personalized consumptions that are possible as a function of the quantity of the shared good that is available. To distinguish between the good that provides the shared benefits and the personalized consumptions obtained from it, he refers to the former as a shared good facility and restricts the use of the term "shared good" to the latter. This terminological practice is also followed here. In effect, a sharing technology is a production function that has shared good facilities as inputs and personalized shared good consumptions as outputs.[2]

This way of modeling impure public goods allows for various degrees of rivalry in consumption. As has been noted, private goods and public goods are special cases. With a private good, the quantities of the personalized consumptions and the shared good facility are measured in the same units with it being possible to distribute the quantity of the shared good facility among the individuals in any way that respects the constraints that the individual consumptions are non-negative and sum to the quantity of the shared good facility available. With a public good, each individual is constrained to

[2]For a discussion of alternative ways in which production theory has been used to model impure public goods, see Weymark (2004, p. 177).

consume the same amount of the good when it is provided efficiently.

Weymark (2004) separates the production of shared good facilities from the sharing of the benefits of these facilities, with the latter carried out by a separate set of agents called groups. He uses his model to identify the necessary conditions for an allocation to be Pareto optimal in a shared goods economy. In the special cases in which the shared goods are private or public goods, these conditions simplify to the standard optimality conditions for private and public goods, respectively.

Weymark's way of modeling shared goods supposes that a shared good facility can only be used for one kind of shared good consumption and that only one shared good facility is needed to obtain the benefits of any particular kind of shared good. Both of these assumptions are overly restrictive and are relaxed here by supposing that a shared good facility may result in multiple kinds of benefits to consumers and that some of these benefits may require more than one shared good facility for their realization. For example, a fire station is often also used to provide paramedic services and fire protection depends not just on having a nearby fire station, but also on the presence of roads for the fire trucks to access the site of a fire.

The proof strategy that is employed to show the existence of a competitive shared goods equilibrium builds on the proofs independently developed by Fabre-Sender (1969) and Foley (1970) for Lindahl equilibria. This proof involves constructing an associated private goods economy in which each person's consumption of a shared good is a separate personalized private good. Here, it is shown that a Walrasian equilibrium exists in this artificial economy if and only if a competitive shared goods equilibrium exists in the actual economy.[3]

The market-based solution proposed by Arrow (1969) to deal with the problems raised by externalities also makes use of personalized private goods. For Arrow, if consumer i's consumption x_{ik} of good k creates a consumption externality, i's consumption can be thought of as being the input in a joint production process whose outputs are the values $x_{hik} = x_{ik}$ for all individuals h. In other words, x_{hik} is the quantity of a personalized private good for h whose value is the quantity of good k consumed by i. If there are complete markets, each of these personalized goods has its own price. In effect, Arrow's

[3]For good introductions to the literature on general equilibrium theory with public goods and to the proof strategies employed by Fabre-Sender and Foley, see Milleron (1972) and Florenzano (2009, 2010).

joint production process is a special kind of sharing technology.

For exchange economies with consumption externalities, Bonnisseau et al. (2023) investigate the existence of equilibria when Arrovian markets with personalized Lindahl prices are used to value the externalities. Using a proof strategy similar in spirit to those used by Fabre-Sender and Foley to show the existence of a Lindahl equilibrium for public goods, they establish the existence of an Arrovian market equilibrium by constructing an associated private goods economy with no externalities that has a production sector which uses the consumptions of private goods as inputs to produce the same personalized allocation for each consumer.

Arrow's market-based approach to resource allocation in the presence of externalities has also been explored in a series of articles by Bergstrom that culminate in Bergstrom (1976). He considers a model with what he calls "communal commodities" that can deal with various kinds of externalities. As is the case with a Lindahl equilibrium for public goods, in Bergstrom's definition of a generalized Lindahl equilibrium, personalized prices are used for the communal commodities. However, the proof of his existence theorem does not employ the methodology used by Fabre-Sender and Foley.

The plan for the rest of this article is as follows. The model of a shared goods economy is introduced in Section 2. The definition of a competitive shared goods equilibrium is provided in Section 3. Alternative strategies for proving the existence of a Lindahl equilibrium are described in Section 4. The associated private goods economy in which the consumptions of shared goods are treated as personalized private goods is constructed in Section 5. In Section 6, it is shown that there are isomorphisms between the shared goods economy and the associated private goods economy and between the competitive shared goods equilibria in the former and the Walrasian equilibria in the latter. In Section 7, the existence of a Walrasian equilibrium in the associated private goods economy is established, from which it follows that there exists a competitive shared goods equilibrium in the shared goods economy. The alternative equilibrium concept in which shared goods are provided cooperatively is considered in Section 8. Section 9 presents some concluding remarks.

2. The Model of the Economy

There are three classes of agents in the economy with a finite number of agents in each class. Consumers are indexed by the set $\mathcal{I} = (1, \ldots, I)$, firms by the set $\mathcal{J} = (1, \ldots, J)$, and groups that provide the shared goods facilities by the set $\mathcal{K} = (1, \ldots, K)$. Groups are interpreted as being coalitions of individuals, with a particular individual possibly belonging to more than one group. The beneficiaries of a group's shared good facility need not be restricted to members of that group, although that is a natural special case to consider.

There are also a finite number of goods. They, too, are partitioned into three categories. Private goods are indexed by $\mathcal{L} = (1, \ldots, L)$, shared goods by $\mathcal{M} = (1, \ldots, M)$, and shared good facilities by $\mathcal{N} = (1, \ldots, N)$. Shared goods are measured in consumption units (e.g., how much fire protection is provided), whereas shared good facilities are measured in production units (e.g., the capacity of a fire station). Because shared goods and shared good facilities need not stand in a one-to-one relationship, it need not be the case that $M = N$. That is the case in the model of shared goods considered in Weymark (2004).

Consumer $i \in \mathcal{I}$ has a *consumption set* $X_i \subseteq \mathbb{R}^{L+M}$. A typical element of X_i is the *consumption bundle*

$$\mathbf{x}_i = (x_{i1}^L, \ldots, x_{iL}^L, x_{i1}^{M_i}, \ldots x_{iM}^{M_i}) = (\mathbf{x}_i^L, \mathbf{x}_i^{M_i}),$$

where the first L components of $\mathbf{x}_i$ are the consumptions of the private goods and the last M are the consumptions of the shared goods.[4] Shared good facilities only enter into the production of shared goods and are, therefore, not part of a consumption bundle.

The economy is one of private ownership. Accordingly, ownership of resources and claims to profits are vested in the individual consumers. There are no endowments of shared goods or shared good facilities. It is convenient to have the dimension of an individuals resource endowment to have the same dimension as a consumption bundle. Person i's *endowment of goods* is

$$\boldsymbol{\omega}_i = (\omega_{i1}^L, \ldots \omega_{iL}^L, \omega_{i1}^{M_i}, \ldots \omega_{iM}^{M_i}) = (\boldsymbol{\omega}_i^L, \mathbf{0}_M) \in \mathbb{R}_+^L \times \mathbf{0}_M.^5$$

For all $i \in \mathcal{I}$, consumer i's *preference* $\succeq_i$ is a binary relation on X_i interpreted as "weakly preferred to." The strict preference and indifference relations $\succ_i$ and $\sim_i$ are defined in the standard fashion. By defining $\succeq_i$ on X_i, it is implicitly being assumed that consumers do not have preferences about

[4]The use of the superscript M_i rather than simply M reflects the personalized nature of shared goods.

[5]For any postive integer $r > 0$, $\mathbf{0}_r$ is a r-dimensional vector of zeros.

the shared good facilities and that there are no consumption externalities except indirectly through the sharing technologies.

The following assumptions are made about the preferences and consumption sets.[6]

Assumption 1. For all $i \in \mathcal{I}$,

(i) X_i is closed, convex, and has a lower bound for $\leq$.

(ii) There exists $\mathbf{x}_i^\circ \in X_i$ such that $\mathbf{x}^{L\circ} \ll \boldsymbol{\omega}_i^L$ and $\mathbf{x}_i^{M_i\circ} = \mathbf{0}_M$.

(iii) $\succeq_i$ is an ordering (reflexive, complete, and transitive).

(iv) $\succeq_i$ is continuous.

(v) $\succeq_i$ exhibits local non-satiation (i.e., for all $\mathbf{x}_i \in X_i$ and all $\varepsilon > 0$, there exists an $\mathbf{x}_i^\circ \in X_i$ in an ε-neighborhood of $\mathbf{x}_i$ such that $\mathbf{x}_i^\circ \succ_i \mathbf{x}_i$).

(vi) $\succeq_i$ is convex (i.e., for all $\mathbf{x}_i \in X_i$,, the set $\{\mathbf{x}'_i \in X_i \mid \mathbf{x}'_i \succeq_i \mathbf{x}_i\}$ is convex).

Assumptions 1.(i), 1.(iii), and 1.(iv) imply that $\succeq_i$ can be represented by a continuous utility function (Debreu, 1959, p. 56). Assumption 1.(i) is satisfied if, as is commonly assumed, X_i is the non-negative orthant. Assumption 1.(ii) requires that it be possible for a consumer to consume less of each private good than he is endowed with and at the same time consume no shared goods.

Firm $j \in \mathcal{J}$ has a *production set* $Y_j \subseteq \mathbb{R}^{L+N}$. A typical element of Y_j is the *production plan*

$$\mathbf{y}_j = (y_{j1}^L, \ldots, y_{jL}^L, y_{j1}^N, \ldots y_{jN}^N) = (\mathbf{y}_j^L, \mathbf{y}_j^N),$$

where the first L components of $\mathbf{y}_j$ are the inputs and outputs of the private goods and the last N are the inputs and outputs of the shared good facilities. Negative values of $\mathbf{y}_j$ correspond to inputs and positive values to outputs. The *aggregate production set* is

$$Y = \sum_{j \in \mathcal{J}} Y_j.$$

The following assumptions are made about the firms' production sets.

[6]For formal definitions of terms that are undefined in this section, see Debreu (1959).

Assumption 2. For all $j \in \mathcal{J}$,

(i) $\mathbf{0}_{L+N} \in Y_j$ (inactivity is possible).

(ii) Y_j is closed and convex.

(iii) $Y_j \cap -Y_j = \{\mathbf{0}_{L+N}\}$ (production is irreversible).

(iv) $\mathbb{R}_-^{L+N} \subseteq Y_j$ (free disposal).[7]

(v) If $(\mathbf{y}_j^L, \mathbf{y}_j^N) \in Y_j$ and for all $n \in \mathcal{N}, \bar{y}_{jn}^N = y_{jn}^N$ when $y_{jn}^N \geq 0$ and $\bar{y}_{jn}^N = 0$ when $y_{jn}^N < 0$, then $(\mathbf{y}_j^L, \bar{\mathbf{y}}_j^N) \in Y_j$ (shared good facilities are inessential).

For firms, the production of a shared good facility is no different in kind from the production of any other good. Except for Assumption 2.(v), these assumptions are familiar assumptions made about firms' technologies. Assumption 2.(v) says that the private goods components of any feasible production plan that makes use of a shared good facility as an input is feasible even if none of this facility is used. As a consequence, provided that the price of a facility is positive, a profit-maximizing firm would not want to use it as an input.[8] An implication of Assumptions 2.(iii) and 2.(iv) is that $Y_j \cap \mathbb{R}_+^{L+N} = \{\mathbf{0}_{L+N}\}$ for all $j \in \mathcal{J}$.

In other respects, the technologies are quite standard. A firm may have multiple inputs and outputs. However, it is only concerned with private goods and shared good facilities.

Assumption 2 applies to all firms. It does not imply that it is possible for any firm to produce a positive amount of any shared good facility. Assumption 3 requires that a positive amount of each shared good facility can be produced by some firm.

Assumption 3. For all shared goods facilities $n \in \mathcal{N}$, there exists a firm $j \in \mathcal{J}$ and a $\mathbf{y}_j \in Y_j$ such that $y_{jn}^N > 0$.

[7]Free disposal may be less justified here than in a private goods economy. It is made for simplicity.

[8]Shared goods facilities are regarded as outputs of the firms' production precesses, not inputs, but in order to allow for free disposal, negative values for the quantities of shared goods facilities are permitted. While this assumption is not necessary for the subsequent analysis, it is made in order to keep the shared or public nature of the problem focused on the consumers and the groups that provide the shared goods. If it is instead assumed that firms can only have shared good facilities as outputs (and not as inputs), then the free disposal assumption (Assumption 2.(iv)) would need to be modified.

The final set of agents are the groups that transform shared good facilities into the personalized consumptions of shared goods. Each group $k \in \mathcal{K}$ has a *sharing technology* $S_k \subseteq \mathbb{R}^{IM+N}$. A typical element of S_k is the *sharing plan*

$$\mathbf{s}_k = (s_{k1}^{M_1}, \ldots, s_{kM}^{M_1}, \ldots, s_{kM}^{M_I}, s_{k1}^{N}, \ldots s_{kN}^{N}) = (\mathbf{s}_k^{M_1}, \ldots, \mathbf{s}_k^{M_I}, \mathbf{s}_k^{N}).$$

In this plan, $s_{km}^{M_i}$ is the consumption of the mth shared good provided by group k to consumer i and s_{kn}^{N} is the quantity of the nth shared good facility used by this group. Because groups use shared goods as inputs, the value of s_{kn}^{N} is non-positive (see Assumption 4). The *aggregate sharing technology* is

$$S = \sum_{k \in \mathcal{K}} S_k.$$

The following assumptions are made about the groups' sharing technologies.

Assumption 4. For all $k \in \mathcal{K}$,

(i) $\mathbf{0}_{IM+N} \in S_k$ (inactivity is possible).

(ii) S_k is closed and convex.

(iii) $S_k \cap -S_k = \{\mathbf{0}_{IM+N}\}$ (sharing is irreversible).

(iv) $\mathbb{R}_{-}^{IM+N} \subseteq S_k$ (free disposal).

(v) If $\mathbf{s}_k \in S_k$ and for all $i \in \mathcal{I}$ and $n \in \mathcal{N}$, $\bar{s}_{kn}^{M_i} = s_{kmn}^{M_i}$ when $s_{kn}^{M_i} \geq 0$ and $\bar{s}_{kn}^{M_i} = 0$ when $s_{kn}^{M_i} < 0$, then $(\bar{\mathbf{s}}_k^{M_1}, \ldots, \bar{\mathbf{s}}_k^{M_I}, \mathbf{s}_k^{N}) \in S_k$ (shared good inputs are inessential).

(vi) For all $n \in \mathcal{N}$, $s_{kn}^{N} \leq 0$ (no shared goods facility outputs).

Assumption 4.(v) formalizes the idea that shared goods are outputs produced for consumption. As is the case with firm production sets, in equilibrium, only non-negative amounts of these goods are part of a sharing plan.[9] Assumption 4.(vi) in conjunction with Assumption 2 implies that shared goods facilities are only produced by firms. The other parts of Assumption 4

[9]If it is assumed that only non-negative quantities of these goods are possible, then the statement of the free disposal assumption would need to be modified. For simplicity, the assumption in the text is made instead.

are standard assumptions on technologies. Analogous to to what is the case for firms, Assumptions 4.(iii) and 4.(iv) imply that $S_k \cap \mathbb{R}_+^{IM+N} = \{\mathbf{0}_{IM+N}\}$ for all $k \in \mathcal{K}$.

Assumption 4 does not imply that it is possible for any group to produce a positive amount of any individual's shared good. Assumption 5 requires that a positive amount of each individual's shared goods can be produced by some group.

Assumption 5. For every consumer $i \in \mathcal{I}$ and shared good $m \in \mathcal{M}$, there exists a group $k \in \mathcal{K}$ and a $\mathbf{s}_k \in S_k$ such that $s_{km}^{M_i} > 0$.

A number of special kinds of sharing technologies of interest satisfy Assumption 4. For example, as in Weymark (2004), each shared goods facility could produce only one kind of shared good. Another possibility is that a group is only able to provide shared goods to its own members. This is the usual assumption made in the theory of clubs. If it is further assumed that the groups form a partition of the set of individuals and that the members of a group all receive the same quantity of the goods they share, then the shared goods are local public goods in the sense of Tiebout (1956) and the groups correspond to his jurisdictions.

Having described the technologies of firms and groups, it is now possible to complete the specification of the individual endowments. In addition to their personalized endowments of private goods, individuals have claims to the net outputs of the firms and groups. In principle, these claims could take two forms. First, as in models of stock market economies (Drèze, 1974), each individual could have a claim to a share of the net outputs of the firms and groups. Because the outputs of groups are personalized consumptions of shared goods, this is not natural when there are shared goods. Second, individual claims on firms and groups could take the form of shares in their profits. This is what is assumed here. Specifically, it is assumed that for all $i \in \mathcal{I}$, i's *endowment of shares* is

$$\boldsymbol{\theta}_i = (\theta_{i1}^J, \ldots, \theta_{iJ}^J, \theta_{i1}^K, \ldots, \theta_{iK}^K) = (\boldsymbol{\theta}_i^J, \boldsymbol{\theta}_i^K) \in \mathbb{R}_+^{J+K}$$

with

$$\sum_{i \in \mathcal{I}} \theta_{ij}^J = 1 \text{ and } \sum_{i \in \mathcal{I}} \theta_{ik}^K = 1 \text{ for all } j \in \mathcal{J} \text{ and all } k \in \mathcal{K}.$$

The first J components of $\boldsymbol{\theta}_i$ are i's shares of the firms' profits and the last K components are his shares of the groups' profits.

It is assumed that each individual has a positive share in the profits of every firm and group.

Assumption 6. For all $i \in \mathcal{I}$, $\boldsymbol{\theta}_i \gg \mathbf{0}_{J+K}$.

This assumption is adopted to help ensure that each individual's wealth is sufficient to purchase the least costly bundle in his consumption set. This is a rather strong assumption that is only employed to avoid the technicalities that arise when the consumption bundle that a consumer demands lies on the lower boundary of his consumption set.

An allocation is a specification of the consumption bundles of each consumer, the production plans of each firm, and the sharing plans of each group. Formally, an *allocation* is a vector

$$\mathbf{a} = (\mathbf{x}_1, \ldots, \mathbf{x}_I, \mathbf{y}_1, \ldots, \mathbf{y}_J, \mathbf{s}_1, \ldots, \mathbf{s}_K).$$

The set of *feasible allocations* $\mathcal{A}$ is the set of all allocations $\mathbf{a}$ that satisfy Conditions 1.

Conditions 1. (i) $\mathbf{x}_i \in X_i$ for all $i \in \mathcal{I}$.

(ii) $\mathbf{y}_i \in Y_j$ for all $j \in \mathcal{J}$.

(iii) $\mathbf{s}_k \in S_k$ for all $k \in \mathcal{K}$.

(iv) $\sum_{i \in \mathcal{I}} x_{il}^L \leq \sum_{j \in \mathcal{J}} y_{jl}^L + \sum_{i \in \mathcal{I}} \omega_{il}^L$ for all $l \in \mathcal{L}$.

(v) $x_{im}^{M_i} \leq \sum_{k \in \mathcal{K}} s_{km}^{M_i}$ for all $i \in \mathcal{I}$ and all $m \in \mathcal{M}$.

(vi) $0 \leq \sum_{j \in \mathcal{J}} y_{jn}^N + \sum_{k \in \mathcal{K}} s_{kn}^N$ for all $n \in \mathcal{N}$.

Conditions 1.(i), 1.(ii), and 1.(iii) restrict allocations to those that satisfy the constraints imposed by the the consumption sets, the production sets, and the sharing technologies. Condition 1.(iv) states the materials balance constraints for the private goods. Condition 1.(v) requires the personalized consumption of each shared good to not exceed the amount of that good produced in the aggregate by the groups. The inequalities in Condition 1.(vi) are the materials balance constraints for the shared goods facilities. Because the second sum in each of these inequalities is non-positive, the first sums are non-negative.

The strategy used to prove the existence of a competitive shared goods equilibrium requires the set of feasible allocations to be compact. To ensure that this is the case, it is assumed that the aggregate production set and the aggregate sharing technology are closed.[10] The *aggregate production set* is

$$Y = \sum_{j \in \mathcal{J}} Y_j$$

and the *aggregate sharing technology* is

$$S = \sum_{k \in \mathcal{K}} S_k.$$

These sets inherit all of the properties listed in Assumptions 2 and 4 for the sets that are used to construct them except for their closure. As is standard in existence of equilibrium proofs, the closure of an aggregate technology is assumed directly.

Assumption 7. (i) Y is closed.

(ii) S is closed.

The economy described in this section supplements the preferences of the consumers and the physical constraints on the consumption, production, and sharing of the private and shared goods with the institutional assumption that resources are privately owned. Formally, a *shared goods private ownership economy* is a tuple

$$\mathcal{E} = (\langle \succeq_i, X_i, \boldsymbol{\omega}_i, \boldsymbol{\theta}_i \rangle_{i \in \mathcal{I}}, \langle Y_j \rangle_{j \in \mathcal{J}}, \langle S_k \rangle_{k \in \mathcal{K}}).$$

Let $\mathcal{D}$ be the ordered pair

$$\mathcal{D} = (\mathcal{E}, \mathcal{A}).$$

3. Competitive Shared Goods Equilibria

An essential feature of competitive markets is the parametric role played by prices. Each agent takes the price of each good as given and makes decisions based on the assumption that any quantity of a good may be bought or sold

[10]The need to make this assumption is due to the fact that the sum of closed sets need not be closed. The proof that the assumptions adopted in this section imply that the set of feasible allocations is compact is omitted. What is important for the equilibirum existence proof is that set of feasible allocations in the associated private goods economy introduced in Section 5 is compact. This issue is addressed in Section 7.

at its market price. In principle, each agent in an economy could have a personalized price for each good. However, restrictions on the nature of a good result in restrictions on the number of different prices that are observed. With a private good and no transactions costs, all agents face the same price for this good. For a pure public good, a Lindahl equilibrium is characterized by personalized prices for consumers but a common selling price for the firms that supply this good. In this case, the personalized buying prices sum to the common selling price.

For the shared goods considered here, as in the traditional theory of public goods, sharing only takes place among consumers. In particular, there are no shared goods inputs. Accordingly, equilibrium prices for a shared good can be personalized but the prices of the private goods and of the shared good facilities must be the same for all of the agents. When a private or public good is thought of as being a special kind of shared good, the restrictions on the prices that hold in these cases are an endogenous feature of the equilibrium analysis rather than being assumed from the outset.

A *price system* is a tuple

$$\mathbf{p} = (\mathbf{p}^L, \mathbf{p}^{M_1}, \ldots, \mathbf{p}^{M_I}, \mathbf{p}^N),$$

where

$$\mathbf{p}^L = (p_1^L, \ldots, p_L^L)$$

are the prices of the private goods,

$$\mathbf{p}^{M_i} = (p_1^{M_i}, \ldots, p_M^{M_i})$$

are the personalized prices for i's consumptions of the shared goods for all $i \in \mathcal{I}$, and

$$\mathbf{p}^N = (p_1^N, \ldots, p_N^N)$$

are the prices of the shared goods facilities.

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $k \in \mathcal{K}$, let

$$\mathbf{p}^i = (\mathbf{p}^L, \mathbf{p}^{M_i}),$$

$$\mathbf{p}^j = (\mathbf{p}^L, \mathbf{p}^N),$$

and

$$\mathbf{p}^k = (\mathbf{p}^{M_1}, \ldots, \mathbf{p}^{M_I}, \mathbf{p}^N)$$

be the prices faced by consumer i, firm j, and group k, respectively.

Each of these price vectors has the same dimension as the corresponding consumption set, production set, or sharing technology. For consumer i, the first L components of $\mathbf{p}^i$ are the prices of the private goods and the last M components are the personalized prices that i faces for the shared goods. For firm j, the first L components of $\mathbf{p}^i$ are the prices of the private goods and the last N components are the prices for the shared goods facilities. For group k, the first IM components of $\mathbf{p}^k$ are the personalized prices for the shared goods and the the last N components are the prices for the shared goods facilities. Note that (i) consumers and firms face the same prices for private goods, (ii) firms and groups face the same prices for shared goods facilities, and (iii) consumers and groups face the same personalized prices for shared goods.

A *competitive shared goods equilibrium* for $\mathcal{D} = (\mathcal{E}, \mathcal{A})$ is a tuple

$$C(\mathcal{D}) = (\mathbf{a}^*, \mathbf{p}^*) = (\langle \mathbf{x}_i^* \rangle_{i \in \mathcal{I}}, \langle \mathbf{y}_j^* \rangle_{j \in \mathcal{J}}, \langle \mathbf{s}_k^* \rangle_{k \in \mathcal{K}}, \mathbf{p}^*)$$

that satisfies Conditions 2.

Conditions 2. (i) $\mathbf{a}^* \in \mathcal{A}$.

(ii) For all $i \in \mathcal{I}$, $\mathbf{x}_i^*$ maximizes $\succeq_i$ on

$$\{\mathbf{x}_i \in X_i \mid \mathbf{p}^{i*}\mathbf{x}_i \leq \mathbf{p}^{i*}\boldsymbol{\omega}_i + \sum_{j \in \mathcal{J}} \theta_{ij}\mathbf{p}^{j*}\mathbf{y}_j^* + \sum_{k \in \mathcal{K}} \theta_{iJ+k}\mathbf{p}^{k*}\mathbf{s}_k^*\}.$$

(iii) For all $j \in \mathcal{J}$, $\mathbf{y}_j^* \in \arg\max_{\mathbf{y}_j \in Y_j} \mathbf{p}^{j*}\mathbf{y}_j$.

(iv) For all $k \in \mathcal{K}$, $\mathbf{s}_k^* \in \arg\max_{\mathbf{s}_k \in S_k} \mathbf{p}^{k*}\mathbf{s}_k$.

Condition 2.(i) requires that an equilibrium allocation be feasible.

Conditions 2.(ii), 2.(iii), and 2.(iv) describe the behavior of consumers, firms, and groups, respectively. Because all markets are competitive, all of these agents take prices as given when making their decisions. Each consumer chooses a most-preferred affordable consumption bundle. That is, he chooses a demand vector. Because his preferences can be represented by a utility function, this amounts to requiring utility maximization subject to a budget constraint. A consumer's income is the sum of the value of his endowment of private goods and his share of the profits of firms and groups. Each firm and group is a profit maximizer.

No *a priori* restrictions have been placed on the signs of the prices. Nevertheless, the assumptions that have been made imply that in a competitive shared goods equilibrium, all prices are non-negative and that at least one of them is positive. Free disposal in production and sharing (Assumptions 2.(iv) and 4.(iv)) rule out negative prices (Debreu, 1959, p. 47).[11] Nonsatiation in preferences (Assumption 1.(v)) excludes the possibility that all prices are free.

Assumption 1 implies that at a solution to a consumer's preference maximization problem, the budget constraint binds. This is the case whether prices are permitted to be negative or not (Debreu, 1959, p. 71).

In his Lindahl equilibrium existence theorem for public goods, rather than assuming that the consumption bundle of a consumer is a demand vector, Foley (1970, p. 70) assumes that any bundle that is not chosen costs more than what is chosen and that the budget constraint holds with equality. The analogous assumption could be made here instead of Condition 2.(ii) because, given Assumption 1, the two variant specifications of this condition result in a consumer choosing the same demands (Debreu, 1959, pp. 68–71).[12]

In the special cases in which shared goods are private or public, a competitive shared goods equilibrium is respectively a Walrasian and a Lindahl equilibrium. When shared goods are private goods, a given quantity of the private good "facility" can be distributed to the consumers in the same way that any private good is. As a consequence, one more unit of the good for one person results in there being one less unit of it for the other consumers. Furthermore, one more unit of the "facility" increases the total consumption of this good by one unit as well. The linearity in the sharing technology combined with the profit maximizing behavior of the groups implies that in equilibrium all agents share a common price for any private good modeled as a shared good. Furthermore, no profits are made by groups supplying private goods. Consequently, a competitive shared goods equilibrium when all of the shared goods are private goods is a Walrasian equilibrium.

For a pure public good, the relevant restriction is also on the sharing technology. In this case, if none of the public (i.e., shared) good is freely disposed, one unit of a public good facility produces one unit of consumption of this public good for each consumer. In other words, the outputs of the tech-

[11] In their existence theorem for private goods, Hart and Kuhn (1975) do not assume free disposal and allow for negative prices. They restrict prices to be on the unit ball.

[12] Although many of the proofs used to establish the existence of a Lindahl equilibrium are adapted from the proof of the existence of a competitive equilibrium for private goods in Debreu (1959), the form in which the assumptions are stated typically does not follow Debreu's example. A notable exception is Fabre-Sender (1969).

nology (the personalized consumptions) are produced in fixed proportions, where the proportions are all equal. Because one person's consumption cannot be traded off for anybody else's consumption (there are kinks on the boundary of the sharing technology), unlike in the case of a private good sharing technology, the shape of the sharing technology does not force the equilibrium consumer prices of a public good to be equal for different consumers. Because all consumers have the same quantity of a public good, the revenue received by the sharing group for each unit of the "facility" that it produces is equal to the sum of the prices that the consumers pay for it. Hence, the sum of the individualized equilibrium prices is equal to the equilibrium price of the "facility." The sharing technology for a public good is a cone and, therefore, a group supplying it has zero profit in equilibrium. Thus, a competitive shared goods equilibrium when all of the shared goods are public goods is a Lindahl equilibrium.

While there are no profits for the groups providing shared goods in the special cases in which the shared goods are private or purely public, a sharing group may have a positive profit in equilibrium when sharing does not take one of these two forms if the group's sharing technology is not a cone.

4. Strategies for Proving the Existence of a Lindahl Equilibrium

The assumptions made about consumers, firms, and groups are sufficient to establish the existence of a competitive shared goods equilibrium. To prove this result, the strategy introduced by Fabre-Sender (1969) and Foley (1970) to prove the existence of a Lindahl equilibrium is generalized so as to apply to a shared goods economy. In this section, their proof strategy is described as well as two alternative strategies that have been used in Lindahl equilibrium existence proofs.[13] Each of these three strategies exploit the fact that all agents face the same prices for private goods and that all consumers consume the same amount of a public good, with these prices and quantities adjusted so as to obtain an equilibrium allocation.

The fundamental insight of Fabre-Sender and Foley was the recognition that the public goods problem can be reformulated as a private goods problem in which each person's consumption of a public good is treated as a separate personalized private good. It is this insight as applied to shared goods that underlies the proof of the existence of a competitive shared goods equilibrium

[13] For further discussion of these proof strategies, see Roberts (1974).

presented here.

The basic idea underlying their proofs is that there exists an isomorphism between an economy with private and public goods and an artificial associated economy with only private goods. Furthermore, a Walrasian equilibrium in the associated economy exists if and only if there is a Lindahl equilibrium in the actual economy.

The isomorphism between the economy with public goods and the associated economy with only private goods is obtained by defining each consumer's consumption of a public good as a separate commodity. Thus, for each public good in the economy there are I private goods in the associated economy—the personalized consumptions of the public good. In any individual's consumption bundle, his consumption of a public good is replaced by his consumption of each of these personalized private goods. Because there are no consumption externalities, each consumer only cares about his own consumption of a public good, so in the associated economy it is necessary that nobody consume anybody else's personalized goods. This is accomplished by assuming that any allocation in an individual's consumption set in the associated economy must have a zero quantity for the consumption of any personalized good that is not his own.

There is a common price for each of the personalized private goods. Thus, the personalized prices for a public good are replaced by person-independent prices for personalized goods in the associated economy. By construction, only one person can buy any of consumer i's personalized goods—person i.

Because everybody consumes the same amount of a public good, it is also necessary that the consumptions of the corresponding personalized goods be the same in the associated economy. This is accomplished by introducing new production sets in the associated economy. These production sets are in effect the special case of a sharing technology described above for public goods.

The properties of the associated economy ensure that there exists a Walrasian equilibrium for it. The isomorphism between the two economies then establishes that a Lindahl equilibrium exists in the economy with public goods.

A second approach to establishing the existence of a Lindahl equilibrium is due to Milleron (1972). His approach exploits the duality between prices and quantities in a private goods economy. Rather than using quantities as the choice variables for consumers who regard prices as exogenous, prices are chosen to support exogenously specified quantities of the goods. The commodity space is expanded as in the approach used by Fabre-Sender and Foley, A mapping is introduced whose fixed points correspond to Lindahl equilibria. In equilibrium, there are agent-independent prices for each of the personalized goods. While Milleron's proof strategy could be generalized so as to apply to shared goods, to exploit the duality between prices and quantities as he does is not straightforward, and so this approach is not pursued here.

Both of the approaches just described require the dimensionality of the commodity space to be expanded. A third strategy for proving the existence of a Lindahl equilibrium is due to Roberts (1974). With his approach, the dimensionality of the commodity space is not modified. Roberts uses the duality highlighted by Millleron but he does not apply it to all goods—private and public. Instead, as in a Walrasian economy, the quantities of the private goods are endogenously chosen given exogenous prices. However, for the public goods, personalized prices are chosen to support quantities of the public goods that are the same for all agents. Instead of establishing the existence of a fixed point in terms of only prices, the mapping used to show the existence of a fixed point is constructed for private goods prices and public goods quantities. This approach does not generalize so as to apply to shared goods (except in the special cases of private and public goods) because they do not fit into one of the two categories of goods used in Roberts' approach.

5. The Associated Private Goods Economy

The proof of the existence of a competitive shared goods equilibrium employs the same basic strategy as in the proofs of the existence of a Lindahl equilibrium used by Fabre-Sender (1969) and Foley (1970) with the sharing technologies playing the role of the artificial production sets that they use for the sharing of a public good. As in their proofs, an associated private goods economy is defined in which each good that is shared is personalized. The set of Walrasian equilibria in the associated economy is shown to be isomorphic to the set of competitive shared goods equilibria in the actual economy. The existence of an equilibrium in the associated economy follows from the equilibrium existence theorem established by Debreu (1959, pp. 83–84) for a competitive private goods economy with private ownership.[14] The existence

[14]Strictly speaking, a minor modification will be made to Debreu's assumptions about the endowments. This modification simplies the specification of the associated economy

of a competitive shared goods equilibrium then follows from the isomorphism between the two kinds of equilibria.

In the rest of the section, the *associated economy* and a Walrasian equilibrium for it are defined.

The commodity space is expanded by including every consumer's personalized consumption of a shared good as a separate good. The consumption bundles of all consumers, the production plans of all firms, and the sharing plans of all groups now include all of these goods. Thus, instead of the $L+M+N$ goods in the shared goods economy, there are $L+IM+N$ of them. These goods are elements of the set

$$\begin{aligned}\hat{Z} &= \{1,\ldots,L,L+1,\ldots,L+M,L+M+1,\ldots,L+IM+N\}\\ &= \{\hat{L},\hat{M}_1,\ldots,\hat{M}_I,\hat{N}\}.\end{aligned}$$

The set $\hat{L}$ indexes the private goods. For all $i \in \mathcal{I}$, the set $\hat{M}_i$ indexes i's shared goods. The set $\hat{N}$ indexes the shared goods facilities.

In the shared goods economy, the dimensions of the consumption sets, production sets, and sharing technologies differ between different kinds of agents. In the associated economy, these sets are all defined on spaces whose dimension is the number of goods in the associated economy, $L+IM+N$. For components that do not appear in the original description of the economy, zeros are inserted. Thus, for all $i \in \mathcal{I}$, the *associated consumption set* is

$$\hat{X}_i = \{\hat{\mathbf{x}}_i \mid \hat{\mathbf{x}}_i = (\mathbf{x}_i^L, \mathbf{0}_{(i-1)M}, \mathbf{x}_i^{M_i}, \mathbf{0}_{N+IM-iM}) \text{ for some } \mathbf{x}_i \in X_i\},$$

for all $j \in \mathcal{J}$, the *associated production set* is

$$\hat{Y}_j = \{\hat{\mathbf{y}}_j \mid \hat{\mathbf{y}}_j = (\mathbf{y}_i^L, \mathbf{0}_{IM}, \mathbf{y}_j^N) \text{ for some } \mathbf{y}_j \in Y_j\},$$

and for all $k \in \mathcal{K}$, the *associated sharing technology* is

$$\hat{S}_k = \{\hat{\mathbf{s}}_k \mid \hat{\mathbf{s}}_k = (\mathbf{0}_L, \mathbf{s}_k) \text{ for some } \mathbf{s}_k \in S_k\}.$$

The *associated aggregate production set* is

$$\hat{Y} = \sum_{j\in\mathcal{J}} \hat{Y}_j$$

---

without changing his existence theorem in any significant way.

and the *associated aggregate sharing technology* is

$$\hat{S} = \sum_{k \in \mathcal{K}} \hat{S}_k.$$

Recall that the only goods that a consumer is endowed with are private goods. Thus, for all $i \in \mathcal{I}$, the *associated endowment of goods* is

$$\hat{\boldsymbol{\omega}}_i = (\boldsymbol{\omega}_i^L, \mathbf{0}_{IM+N}).$$

Endowments of the shares in firms and in groups are the same in both the shared goods and the associated economies. Thus, for all $i \in \mathcal{I}$, the *associated endowment of shares* is

$$\hat{\boldsymbol{\theta}}_i = \boldsymbol{\theta}_i.$$

For all $i \in \mathcal{I}$, consumer i's *associated preference* $\hat{\succeq}_i$ is defined in the natural way on his associated consumption set. Let $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{x}}_i^\circ$ be the associated consumption bundles in $\hat{X}_i$ constructed from the consumption bundles $\mathbf{x}_i$ and $\mathbf{x}_i^\circ$ in X_i. Then,

$$\hat{\mathbf{x}}_i \hat{\succeq}_i \hat{\mathbf{x}}_i^\circ \leftrightarrow \mathbf{x}_i \succeq_i \mathbf{x}_i^\circ.$$

An *associated allocation* is a vector

$$\hat{\mathbf{a}} = (\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_I, \hat{\mathbf{y}}_1, \ldots, \hat{\mathbf{y}}_J, \hat{\mathbf{s}}_1, \ldots, \hat{\mathbf{s}}_K).$$

The set of *feasible associated allocations* $\hat{\mathcal{A}}$ is the set of associated allocations that satisfy Conditions $1'$.

Conditions $1'$. (i) $\hat{\mathbf{x}}_i \in \hat{X}_i$ for all $i \in \mathcal{I}$.

(ii) $\hat{\mathbf{y}}_i \in \hat{Y}_j$ for all $j \in \mathcal{J}$.

(iii) $\hat{\mathbf{s}}_k \in \hat{S}_k$ for all $k \in \mathcal{K}$.

(iv) $\sum_{i \in \mathcal{I}} \hat{x}_{iz} \leq \sum_{j \in \mathcal{J}} \hat{y}_{jz} + \sum_{k \in \mathcal{K}} \hat{s}_{kz} + \sum_{i \in \mathcal{I}} \hat{\omega}_{iz}$ for all $z \in \mathcal{Z}$.

Note that by expanding the dimensionality of the commodity space, Conditions 1.(iv), 1.(v), and 1.(vi) that are used to express the materials balance constraints for the private goods, shared goods, and shared goods facilities in the shared goods economy can be replaced by a single condition, Condition $1'$.(iv), in the associated economy.

The *associated shared goods private ownership economy* is a tuple

$$\hat{\mathcal{E}} = (\langle \hat{\succeq}_i, \hat{X}_i, \hat{\boldsymbol{\omega}}_i, \hat{\boldsymbol{\theta}}_i \rangle_{i \in \mathcal{I}}, \langle \hat{Y}_j \rangle_{j \in \mathcal{J}}, \langle \hat{S}_k \rangle_{jk \in \mathcal{K}}).$$

Let $\hat{\mathcal{D}}$ be the ordered pair

$$\hat{\mathcal{D}} = (\hat{\mathcal{E}}, \hat{\mathcal{A}}).$$

In the associated economy, all goods, even the personalized ones, are private goods. The equilibrium concept for this economy is the Walrasian one. Accordingly, the price of each good is agent independent. The *associated prices* are

$$\hat{\mathbf{p}} = (\hat{p}_1, \ldots, \hat{p}_{L+IM+N}).$$

In this price vector, the prices are listed in the same order as the goods are ordered in the allocations in $\hat{Z}$.

A *Walrasian equilibrium* for $\hat{\mathcal{D}} = (\hat{E}, \hat{\mathcal{A}})$ is a tuple

$$\hat{C}(\hat{\mathcal{D}}) = (\hat{\mathbf{a}}^*, \hat{\mathbf{p}}^*) = (\langle \hat{\mathbf{x}}_i^* \rangle_{i \in \mathcal{I}}, \langle \hat{\mathbf{y}}_j^* \rangle_{j \in \mathcal{J}}, \langle \hat{\mathbf{s}}_k^* \rangle_{k \in \mathcal{K}}, \langle \hat{\mathbf{p}}_z^* \rangle_{z \in \mathcal{Z}})$$

that satisfies Conditions $2'$.

Conditions $2'$. (i) $\hat{\mathbf{a}}^* \in \hat{\mathcal{A}}$.

(ii) For all $i \in \mathcal{I}$, $\hat{\mathbf{x}}_i^*$ maximizes $\hat{\succeq}_i$ on

$$\{\hat{\mathbf{x}}_i \in \hat{X}_i \mid \hat{\mathbf{p}}^* \hat{\mathbf{x}}_i \leq \hat{\mathbf{p}}^* \hat{\boldsymbol{\omega}}_i + \sum_{j \in \mathcal{J}} \hat{\theta}_{ij} \hat{\mathbf{p}}^* \hat{\mathbf{y}}_j^* + \sum_{k \in \mathcal{K}} \hat{\theta}_{iJ+k} \hat{\mathbf{p}}^* \hat{\mathbf{s}}_k^* \}.$$

(iii) For all $j \in \mathcal{J}$, $\hat{\mathbf{y}}_j^* \in \arg\max_{\hat{\mathbf{y}}_j \in \hat{Y}_j} \hat{\mathbf{p}}^* \hat{\mathbf{y}}_j$.

(iv) For all $k \in \mathcal{K}$, $\hat{\mathbf{s}}_k^* \in \arg\max_{\hat{\mathbf{s}}_k \in \hat{S}_k} \hat{\mathbf{p}}^* \hat{\mathbf{s}}_k$.

In other words, in a Walrasian equilibrium for the associated economy, the allocation must be feasible, each consumer chooses a consumption bundle to maximize his preferences at the equilibrium prices, and each firm and group is profit maximizing at these prices.

6. Isomorphisms Between the Economies and Their Equilibria

As has been noted, the proof of the existence of a competitive shared goods equilibrium is established by showing that there exists a Walrasian equilibrium in the associated economy. In order to do so, it is first shown that these two economies and their equilibria are isomorphic to each other. This is done by showing that there exist bijections between the economies $\mathcal{E}$ and $\hat{\mathcal{E}}$, the feasible allocations $\mathcal{A}$ and $\hat{\mathcal{A}}$, and the equilibria $C(\mathcal{E}, \mathcal{A})$ and $\hat{C}(\hat{\mathcal{E}}, \hat{\mathcal{A}})$.

Theorem 1. *There exists a bijection $f\colon \mathcal{D} \to \hat{\mathcal{D}}$, where $f = (f_1, f_2)$, f_1 is a bijection from $\mathcal{E}$ to $\hat{\mathcal{E}}$, and f_2 is a bijection from $\mathcal{A}$ to $\hat{\mathcal{A}}$.*

Proof. The existence of the bijection f_1 follows from the definitions in Section 5.

The function f_1 maps each component of $\mathcal{E}$ into the corresponding component of $\hat{\mathcal{E}}$. In particular, it maps an allocation $\mathbf{a} \in \mathcal{E}$ into the allocation $\hat{\mathbf{a}} \in \hat{\mathcal{E}}$ obtained by (1) for each consumer, adding components whose values are all 0 for the shared goods facilities and the consumptions of the shared goods of the other consumers, (2) for each firm, adding components whose values are all 0 for the consumptions of the shared goods for each consumer, and (3) for each group, adding components whose values are all 0 for the private goods. Let f_2 denote this mapping restricted to the set of feasible allocations $\mathcal{A}$. Because f_1 is a function from $\mathcal{E}$ to $\hat{\mathcal{E}}$ and $\mathcal{A}$ satisfies Conditions 1, using the definitions of $\hat{X}_i$, $\hat{Y}_j$, and $\hat{S}_k$, it follows that $f_2(\mathcal{A})$ satisfies Condition $1'$. Consequently, $f_2(\mathcal{A}) = \hat{\mathcal{A}}$, so f_2 is a function from $\mathcal{A}$ to $\hat{\mathcal{A}}$.

Now, consider the inverse mapping $f_2^{-1}\colon \hat{\mathcal{A}} \to \mathcal{A}$. For an allocation $\hat{\mathbf{a}} \in \hat{\mathcal{A}}$, f_2^{-1} identifies a unique allocation $\mathbf{a} \in \mathcal{A}$ by omitting these components from the assignment made to each agent that do not concern this agent in the shared goods economy. Thus, f_2 is a bijection. Because $\hat{\mathcal{A}}$ satisfies Conditions $1'$.(i), $1'$.(ii), and $1'$.(iii), reasoning as above, $f_2^{-1}(\hat{\mathcal{A}})$ satisfies Conditions 1.(i), 1.(ii), and 1.(iii). The feasible allocations $\hat{\mathcal{A}}$ also satisfy the materials balance constraints in Condition $1'$.(iv). For $l \in \hat{L}$, $\hat{s}_{kl} = 0$. Thus, the materials balance constraints in the associated economy for what are private goods in the shared goods economy simplify to

$$\sum_{i\in\mathcal{I}} \hat{x}_{il} \leq \sum_{j\in\mathcal{J}} \hat{y}_{jl} + \sum_{i\in\mathcal{I}} \hat{\omega}_{il} \text{ for all } l \in \hat{\mathcal{L}}.$$

The components of f_2^{-1} for these goods are identity functions and $\hat{L} = L$, so this equation can be rewritten as

$$\sum_{i\in\mathcal{I}} x_{il}^L \leq \sum_{j\in\mathcal{J}} y_{jl}^L + \sum_{i\in\mathcal{I}} \omega_{il}^L \text{ for all } l \in \mathcal{L},$$

which are the materials balance constraints in Condition 1.(iv). Similar reasoning can be used to show that Condition $1'$.(iv) implies that $f_2^{-1}(\hat{\mathcal{A}})$ satisfies the materials balance constraints in Conditions 1.(v) and 1.(vi). □

Theorem 1 is now used to show that there is an isomorphism between the equilibria in the two economies.

Theorem 2. *There exists a bijection* $g\colon C(\mathcal{D}) \to \hat{C}(\hat{\mathcal{D}})$.

Proof. The proof proceeds by showing that there is an isomorphism between each part of Conditions 2 and 2′.

The isomorphism between allocations in the two economies established in Theorem 1 also establishes the isomorphism between Conditions 2.(i) and 2′.(i).

Because the prices of the private goods and shared goods facilities are the same for all agents in both economies and because the prices of shared goods are personalized in the shared goods economy, there is an isomorphism between the price vectors in these economies as well.

In $\mathcal{D}$, suppose that with the price vector $\mathbf{p}^*$, y_j^* maximizes firm j's profits at the prices $\mathbf{p}^{j*}$. The prices of the private goods and shared goods facilities are the same in the corresponding price vector $\hat{\mathbf{p}}$ in the associated economy $\hat{\mathcal{D}}$. Because $\hat{Y}_j$ is obtained from Y_j by adding components for the shared goods consumptions whose values are all 0, the production plan $\hat{y}_j^*$ obtained from y_j^* by adding a 0 for each of these components maximizes this firm's profits at the prices $\hat{\mathbf{p}}^*$ on $\hat{Y}_j$. The reverse implication is established by removing these components from a firm's production plan. The preceding argument establishes the isomorphism between Conditions 2.(iii) and 2′.(iii).

Analogous reasoning can be used to establish the isomorphism between Conditions 2.(iv) and 2′.(iv). In this case, it is the private goods that are either added or removed from a sharing plan.

In light of the isomorphisms established for the profit-maximizing decisions of the firms and groups, their profits are the same in the allocations $\mathbf{a}$ and $\hat{\mathbf{a}}$ that are obtained from each other as described in the proof of Theorem 1. This observation and the reasoning used above for firms and groups but with it now being the shared goods facilities that are added or removed form a consumption bundle establishes the isomorphism between Conditions 2.(ii) and 2′.(ii). □

The following assumptions about the consumption sets, production sets, sharing technologies, and ownership shares in the associated economy correspond to their analogues in the shared goods economy.

Assumption 1′. For all $i \in \mathcal{I}$,

(i) $\hat{X}_i$ is closed, convex, and has a lower bound for $\leq$.

(ii) There exists $\hat{\mathbf{x}}_i^\circ \in \hat{X}_i$ such that $\hat{x}_i^L \ll \hat{\omega}_i^L$ and $\hat{x}_{ih}^\circ = 0$ for all $h \notin \hat{L}$.

(iii) $\hat{\succeq}_i$ is an ordering.

(iv) $\hat{\succeq}_i$ is continuous.

(v) $\hat{\succeq}_i$ exhibits local non-satiation.

(vi) $\hat{\succeq}_i$ is convex.

Assumption 2′. For all $j \in \mathcal{J}$,

(i) $\mathbf{0}_{L+IM+N} \in \hat{Y}_j$ (inactivity is possible).

(ii) $\hat{Y}_j$ is closed and convex.

(iii) $\hat{Y}_j \cap -\hat{Y}_j = \{\mathbf{0}_{L+IM+N}\}$ (production is irreversible).

(iv) $\mathbb{R}_-^{L+IM+N} \subseteq \hat{Y}_j$ (free disposal).

(v) If $(\hat{\mathbf{y}}_j^L, \mathbf{0}_{IM}, \hat{\mathbf{y}}_j^N) \in \hat{Y}_j$ and for all $n \in \mathcal{N}, \bar{y}_{jL+IM+n} = \hat{y}_{jL+IM+n}$ when $\hat{y}_{jL+IM+n} \geq 0$ and $\bar{y}_{jL+IM+n} = 0$ when $\hat{y}_{jL+IM+n} < 0$, then $(\mathbf{y}_j^L, \mathbf{0}_{IM}, \bar{\mathbf{y}}_j^N) \in \hat{Y}_j$ (shared good facilities are inessential).

Assumption 3′. For all shared goods facilities $n \in \mathcal{N}$, there exists a firm $j \in \mathcal{J}$ and a $\hat{\mathbf{y}}_j \in \hat{Y}_j$ such that $\hat{y}_{jL+IM+n} > 0$.

Assumption 4′. For all $k \in \mathcal{K}$,

(i) $\mathbf{0}_{L+IM+N} \in \hat{S}_j$ (inactivity is possible).

(ii) $\hat{S}_k$ is closed and convex.

(iii) $\hat{S}_k \cap -\hat{S}_k = \{\mathbf{0}_{L+IM+N}\}$ (sharing is irreversible).

(iv) $\mathbb{R}_-^{L+IM+N} \subseteq S_k$ (free disposal).

(v) If $\hat{\mathbf{s}}_k \in \hat{S}_k$ and for all $i \in \mathcal{I}$ and $m \in \mathcal{M}, \bar{s}_{kL+(i-1)M+m} = \hat{s}_{kL+(i-1)M+m}$ when $\hat{s}_{kL+(i-1)M+m} \geq 0$ and $\bar{s}_{kL+(i-1)M+m} = 0$ when $\hat{s}_{kL+(i-1)M+m} < 0$, then $(\mathbf{0}_L, \bar{\mathbf{s}}_k^{M_i}, \ldots, \bar{\mathbf{s}}_k^{M_i}, \hat{\mathbf{s}}_k^N) \in \hat{S}_k$ (shared good inputs are inessential).

(vi) For all $n \in \mathcal{N}, \hat{s}_{kL+IM+n} \leq 0$ (no shared goods facility outputs).

Assumption 5′. For every consumer $i \in \mathcal{I}$ and shared good $m \in \mathcal{M}$, there exists a group $k \in \mathcal{K}$ and a $\hat{\mathbf{s}}_k \in \hat{S}_k$ such that $\hat{s}_{kL+(i-1)M+m} > 0$.

Assumption 6′. For all $i \in \mathcal{I}$, $\hat{\boldsymbol{\theta}}_i \gg \mathbf{0}_{J+K}$.

Assumption 7′. (i) $\hat{Y}$ is closed.

(ii) $\hat{S}$ is closed.

It is straightforward to verify that Assumptions 1–7 and Assumptions 1′–7′ are isomorphic.

7. Existence of a Competitive Shared Goods Equilibrium

In this section, it is shown that the assumptions made about the shared goods economy are sufficient for the existence of a competitive shared goods equilibrium. This is done by showing that a Walrasian equilibrium exists in the associated economy.

One step in Debreu's proof of the existence of a Walrasian equilibrium involves showing that the set of feasible allocations is compact. This proof makes use of properties of his aggregate production set. Here, the analogous set is $\hat{Y} + \hat{S}$. Theorem 3 makes use of Debreu's proof to show that the set of feasible allocations $\hat{A}$ is compact.

Theorem 3. *If the associated economy $\hat{\mathcal{E}}$ satisfies Assumptions 1′, 2′, 4′, and 7′, then the set of feasible allocations $\hat{\mathcal{A}}$ in $\hat{\mathcal{E}}$ is compact and convex.*

Proof. The arguments that Debreu (1959, pp. 76–78) uses to prove the compactness and convexity of the set of feasible allocations in his economy apply equally well to the economy $\hat{\mathcal{E}}$ using the latter assumptions except for showing that $\hat{Y} + \hat{S}$ is closed.

The proof that $\hat{Y} + \hat{S}$ is closed makes use of asymptotic cones. Informally, the *asymptotic cone* F^A of a subset $F \in \mathbb{R}^n$ is the minimal closed cone with a vertex at the origin containing all points in F infinitely far from the origin. The sum of two sets F_1 and F_2 contained in $\mathbb{R}^n$ is closed if $F_1^A \cap F_2^A = \{\mathbf{0}_n\}$.[15] Thus, it is sufficient to show that $\hat{Y}^A \cap \hat{S}^A = \{\mathbf{0}_{L+IM+N}\}$.

Let $\hat{Y}_B$ and $\hat{S}_B$ be the projections of $\hat{Y}$ and $\hat{S}$ to the Euclidean space indexed by the values in $B \subseteq \hat{Z}$. By construction, $\hat{S}_{\hat{L}} = \{\mathbf{0}_L\}$ and $\hat{Y}_{\hat{M}_i} =$

[15] See Debreu (1959, pp. 22–24) for a more formal definition of an asymptotic cone, the result about the closure of a sum of sets, and the results about asymptotic cones appealed to in this proof.

$\{\mathbf{0}_M\}$ for all $i \in \mathcal{I}$. Therefore, it is only necessary to show that $\hat{Y}^A_{\hat{N}} \cap (-\hat{S}^A_{\hat{N}}) = \{\mathbf{0}_N\}$. Because $\hat{Y}_{\hat{N}}$ is closed, convex, and contains the origin, $\hat{Y}^A_{\hat{N}} \subseteq \hat{Y}_{\hat{N}}$. Similarly, $\hat{S}^A_{\hat{N}} \subseteq \hat{S}_{\hat{N}}$. Assumptions 4′.(iv) and 4′.(vi) imply that $\hat{S}_{\hat{N}} = \mathbb{R}^N_-$ and, hence, that $-\hat{S}^A_{\hat{N}} \subseteq \mathbb{R}^N_-$. Assumptions 2′.(iii) and 2′.(iv) imply that $\hat{Y}_{\hat{N}} \cap \mathbb{R}^N_+ = \{\mathbf{0}_N\}$. Hence, because $\{\mathbf{0}_N\} \subseteq \hat{Y}^A_{\hat{N}}$ and $\hat{Y}^A_{\hat{N}} \subseteq \hat{Y}_{\hat{N}}$, $\hat{Y}^A_{\hat{N}} \cap \mathbb{R}^N_+ = \{\mathbf{0}_N\}$. Therefore, $\hat{Y}^A \cap \hat{S}^A = \{\mathbf{0}_{L+IM+N}\}$. □

To prove the existence of a Walrasian equilibrium in a private goods economy, Debreu (1959) assumes that each consumer has a positive endowment of every good and that it is possible to consume less of each good than he is endowed with. With public or shared goods, it is inappropriate to assume that an individual has endowments of these goods (personalized or not). Indeed, Fabre-Sender (1969, p. 37) says that it is "economically absurd" to suppose that someone has a positive endowment of any other individual's personalized consumption of a public good. Because Debreu's assumption about endowments is not satisfied in the associated economy, it is not possible to simply apply his theorem about the existence of a Walrasian equilibrium to the associated economy without modification.

Debreu's assumption about the the endowments of goods was made to ensure that for no vector of semi-positive prices would a consumer's wealth (the sum of the value of his endowments of goods and his profit income) be equal to the value of the cheapest consumption bundle in his consumption set.[16] By ensuring that this is not the case, Debreu's assumptions about the economy imply that each consumer's budget correspondence is continuous in prices when consumptions are bounded from above in a particular way. This continuity result is used to show that that utility maximizing consumptions bundles are correctly identified when some prices are zero and it is needed to apply the fixed point theorem that Debreu employs to prove his existence theorem. Nevertheless, as shall be shown, Debreu's result about the value of a consumer's wealth holds with the weaker assumption made about the endowments of goods made here and, consequently, appropriately restricted budget correspondences in the associated economy can be shown to be continuous.

It is instructive to consider how the literature on the existence of a Lindahl equilibrium has handled this technical difficulty. Foley (1970, p. 70)

[16] A price vector is *semi-positive* if all prices are non-negative and at least one is positive.

applies the Walrasian equilibrium existence theorem in Debreu (1959) to an associated economy with personalized consumptions of the public goods. However, as Milleron (1972, p. 441) points out, Foley's Lemma does not establish what is needed to appeal to Debreu's result about the upper semicontinuity of demand on the boundary of the price simplex. Milleron (1972, pp. 442–448) shows that this difficulty can be resolved by appealing to the more general existence theorem in Debreu (1962). The arguments used to show that Foley's associated economy satisfies the assumptions of Debreu's 1962 existence theorem are quite complex. As shall be argued below, a fairly simple modification of Foley's arguments would have permitted him to appeal to Debreu's 1959 theorem and at the same time avoid the problem that Milleron identified.[17]

Fabre-Sender (1969) addresses this technical issue head on. She makes assumptions that allow Debreu's 1959 theorem to be used in her Lindahl equilibrium existence proof. In particular, her assumptions guarantee that wealth is never minimal and, hence, that the relevant truncated budget correspondence is continuous in prices. This is done by, in effect, assuming that a consumer's profit income is positive whenever the value of the commodity endowment is minimal.[18] In other words, she assumes that this is the case rather than showing that this conclusion follows from assumptions about the primitives of her model. In contrast, here this conclusion is shown to follow from the assumptions made about the model's primitives.[19]

An implication of Assumptions $1'$.(v), $2'$.(iv), and $4'$.(iv) is that Walrasian equilibrium prices in the associated economy must be semi-positive. Theorem 3 has shown the set of feasible allocations $\hat{\mathcal{A}}$ is compact and convex. Thus, for all $i \in \mathcal{I}$, the set of consumption bundles that can be part of a feasible allocation in $\hat{\mathcal{E}}$ must be a compact, convex set $\hat{X}_i^\circ$. For all $i \in \mathcal{I}$, let the *truncated consumption set* $\hat{X}_i'$ be a compact, convex subset of $\hat{X}_i$ that

[17]Foley (1970) models production using an aggregate production set that is assumed to be a cone. There are therefore no profits and, hence, consumers have no profit income. Hence, there is no scope for using profit income to compensate for having commodity endowments whose value is minimal.

[18]A statement of these assumptions and a discussion of their significance may be found in Fabre-Sender (1969, p. 33, pp. 36–37, and Appendix, pp. 12–14).

[19]The proof of this claim is provided in Theorem 4 below. Fabre-Sender permits public goods to be used as inputs in production and does not assume that the endowments of private goods are in the interior of the projection of the consumption set to the subspace that contains only private goods. As a consequence, the strategy used to prove Theorem 4 cannot be used by her.

contains $\hat{X}_i^\circ$ in its interior. As in Debreu (1959, pp. 76–78), for the existence proof, consumer i can be restricted to choosing consumptions bundles in $\hat{X}_i'$. It suffices to show that consumer i's wealth is never minimal on $\hat{X}_i'$ when prices are semi-positive in order to prove that i's budget correspondence is continuous when his demands must be in $\hat{X}_i'$.

Let

$$\hat{\mathbb{P}} = \{\hat{\mathbf{p}} \in \mathbb{R}_+^{L+IM+M} \setminus \{\mathbf{0}_{L+IM+M}\}\}$$

be the set of semi-positive price vectors in the associated economy $\hat{\mathcal{E}}$. For all $j \in \mathcal{J}$ and $k \in \mathcal{K}$, let $\hat{\pi}_j(\hat{\mathbf{p}})$ and $\hat{\pi}_k(\hat{\mathbf{p}})$ be the profits of firm j and group k, respectively, when they maximize profits in $\hat{\mathcal{E}}$ at the prices $\hat{\mathbf{p}} \in \hat{\mathbb{P}}$. Theorem 4 shows that Assumptions 1′–6′ are sufficient to ensure that for each consumer $i \in \mathcal{I}$ and any $\hat{\mathbf{p}} \in \hat{\mathbb{P}}$, i's wealth (from all sources) is not minimal in his truncated consumption set when the firms and groups profit maximize.

Theorem 4. *If the economy $\hat{\mathcal{E}}$ satisfies Assumptions 1′–7′, then for all $i \in \mathcal{I}$ and all $\hat{\mathbf{p}} \in \hat{\mathbb{P}}$,*

$$\hat{\mathbf{p}}\hat{\boldsymbol{\omega}}_i + \sum_{j \in \mathcal{J}} \hat{\boldsymbol{\theta}}_{ij}\hat{\pi}_j(\hat{\mathbf{p}}) + \sum_{k \in \mathcal{K}} \hat{\boldsymbol{\theta}}_{iJ+k}\hat{\pi}_k(\hat{\mathbf{p}}) \neq \min_{\hat{\mathbf{x}}_i \in \hat{X}_i'} \hat{\mathbf{p}}\hat{\mathbf{x}}_i$$

Proof. There are three cases to consider: (1) there exists a $z \in \hat{L}$ such that $\hat{p}_z > 0$; (2) $\hat{p}_z = 0$ for all $z \notin \hat{N}$ and there exists a $z \in \hat{N}$ such that $\hat{p}_z > 0$; and (3) $\hat{p}_z = 0$ for all $z \in \hat{L} \cup \hat{N}$ and there exists an $i' \in \mathcal{I}$ and a $z \in \hat{M}_{i'}$ such that $\hat{p}_z > 0$.

In each of the three cases, Assumptions 2′.(i), 4′.(i), and 6′ ensure that consumer i's profit income is non-negative.

Case (1). Because consumer i's profit income is non-negative, Assumption 1′.(ii) implies that his wealth exceeds the value of $\hat{\mathbf{x}}_i^\circ \in \hat{X}_i'$.

Case (2). In this case, income from the endowments of goods is zero. By Assumptions 2′.(v) and 3′, there must be some firm whose profit is positive when it maximizes profits at these prices. Because all firms and groups make non-negative profits, it then follows from Assumption 6′ that i's wealth is positive. Because $\hat{p}_z = 0$ for all $z \in \hat{L}$, Assumption 1′.(ii) implies that $\hat{\mathbf{p}}\hat{\mathbf{x}}_i^\circ = 0$. Thus, i's wealth is not minimal in $\hat{\mathbf{X}}_i$ at these prices.

Case (3). The proof for this case is the same as in Case (2) except that Assumptions 4′.(v) and 5′ are used to show that some group has a positive profit.[20] □

[20] The proof strategy employed here is, in part, based on arguments used by Foley

With Theorems 3 and 4 in hand, the existence of a competitive shared goods equilibrium in the associated economy follows with minor modifications from the arguments used by Debreu (1959) to establish the existence of a Walrasian equilibrium in his private goods economy.

Theorem 5. *If the associated economy $\hat{\mathcal{E}}$ satisfies Assumptions 1′–7′, then a Walrasian equilibrium $\hat{C}(\hat{\mathcal{D}})$ exists for $\hat{\mathcal{E}}$.*

Proof. All of the arguments used by Debreu (1959, pp. 83–88) to establish the existence of a Walrasian equilibrium apply to the associated economy $\hat{\mathcal{E}}$ when Assumptions 1′–7′ are satisfied except for those that depend on his assumptions that the aggregate technology set is closed and that each consumer's endowment of goods is in the interior in his consumption set. The closure of the aggregate technology needed to apply his theorem is provided by Theorem 3. Debreu's assumption about the endowments is used to show that no matter what the prices are, a consumer's wealth exceeds the value of the cheapest consumption bundle in his consumption set. This result is established here by Theorem 4. □

In view of the isomorphisms established in Section 6, the existence of a competitive shared goods equilibrium in the shared goods economy $\mathcal{E}$ follows immediately from Theorem 5.

Theorem 6. *If the economy $\mathcal{E}$ satisfies Assumptions 1–7, then a competitive shared goods equilibrium $C(\mathcal{D})$ exists for $\mathcal{E}$.*

8. A Cooperative Shared Goods Equilibrium

The concept of a competitive shared goods equilibrium is a natural generalization of a Lindahl equilibrium which, in turn, is a natural public good analogue of a Walrasian equilibrium. With these three kinds of equilibria,

(1967, pp. 62–63; 1970, pp. 68–69). The problem that Milleron pointed out with the use of Debreu's Walrasian equilibrium existence theorem in Foley (1970) can be dealt with using a variant of the proof of Theorem 4. Foley assumes that production sets are cones, so there are no profits. He also assumes that preferences are monotonic, which implies that the equilibrium prices for the private goods must be semi-positive. If, as is the case here, it is assumed that consumer i's endowment of private goods vector dominates the consumptions of these goods in some consumption bundle in his consumption set, then the value of his endowment is never minimal in his consumption set.

each agent takes prices as given for all goods—private, public, or shared—and acts as if he can buy or sell these goods as if they were all private goods.

In this section, a different way of allocating goods is considered in which the groups that provide shared goods are not profit maximizers but, instead, behave cooperatively towards the beneficiaries of their decisions. Specifically, each group proposes a sharing plan, buys shared goods facilities on competitive markets, and charges each consumer a lump-sum amount for the shared goods that it provides. The sharing plan of a group and its accompanying lump-sum charges are chosen so that no other feasible proposal is preferred to it by at least one consumer without harming any other consumer. Given the sharing plans of the groups, each consumer chooses quantities of the private goods to maximize his utility subject to his budget constraint. The amount available to be spent on the private goods is the value of his resource endowment and his profit income less the lump-sum payments made to the groups for the shared goods. Each firm is a competitive profit maximizer. The profits of firms and groups are distributed to consumers according to fixed shares as in a competitive shared goods equilibrium. This kind of equilibrium is called a *cooperative shared goods equilibrium.* It is a shared goods analogue to the concept of a public competitive equilibrium for public goods proposed by Foley (1967, 1970).

Because only the transactions involving private goods and shared goods facilities are mediated using prices, the earlier definition of a price system needs to be modified by removing the prices for the consumptions of shared goods. A *marketed goods price system* is a tuple

$$\tilde{\mathbf{p}} = (\langle \tilde{\mathbf{p}}^i \rangle_{i \in \mathcal{I}}, \langle \tilde{\mathbf{p}}^j \rangle_{j \in \mathcal{J}}, \langle \tilde{\mathbf{p}}^k \rangle_{k \in \mathcal{K}}),$$

where the prices faced by consumer i are

$$\tilde{\mathbf{p}}^i = (\tilde{p}_1, \ldots, \tilde{p}_L) = \tilde{\mathbf{p}}^L,$$

the prices faced by firm j are

$$\tilde{\mathbf{p}}^j = (\tilde{p}_1, \ldots, \tilde{p}_L, \tilde{p}_{L+1}, \ldots, \tilde{p}_{L+N})) = (\tilde{\mathbf{p}}^L, \tilde{\mathbf{p}}^N),$$

and the prices faced by group k are

$$\tilde{\mathbf{p}}^k = (\tilde{p}_{L+1}, \ldots, \tilde{p}_{L+N}) = \tilde{\mathbf{p}}^N.$$

As in the earlier definition of a price system, (i) consumers and firms face the same prices for private goods and (ii) firms and groups face the same prices for shared goods facilities.

A *system of shared goods lump-sum charges* is a tuple

$$\tilde{\boldsymbol{\tau}} = (\tilde{\tau}_1^1, \ldots, \tilde{\tau}_I^1, \ldots, \tilde{\tau}_1^K, \ldots, \tilde{\tau}_I^K),$$

where $\tilde{\tau}_i^k$ is the lump-sum payment to the kth group from the ith consumer for his consumption of his shared goods supplied by this group.

A *cooperative shared goods equilibrium* for the economy $\mathcal{D} = (\mathcal{E}, \mathcal{A})$ is a tuple

$$\begin{aligned}\tilde{C}(\mathcal{D}) &= (\tilde{\mathbf{a}}^*, \tilde{\mathbf{p}}^*, \tilde{\boldsymbol{\tau}}^*) \\ &= (\langle \tilde{\mathbf{x}}_i^* \rangle_{i\in\mathcal{I}}, \langle \tilde{\mathbf{y}}_j^* \rangle_{j\in\mathcal{J}}, \langle \tilde{\mathbf{s}}_k^* \rangle_{k\in\mathcal{K}}, \langle \tilde{\mathbf{p}}^{i*} \rangle_{i\in\mathcal{I}}, \langle \tilde{\mathbf{p}}^{j*} \rangle_{j\in\mathcal{J}}, \langle \tilde{\mathbf{p}}^{k*} \rangle_{k\in\mathcal{K}}, \tilde{\boldsymbol{\tau}}^*)\end{aligned}$$

that satisfies Conditions $2''$.

Conditions $2''$. (i) $\tilde{\mathbf{a}}^* \in \mathcal{A}$.

(ii) For all $i \in \mathcal{I}$, $\tilde{\mathbf{x}}_i^* = (\tilde{\mathbf{x}}_i^{L*}, \tilde{\mathbf{x}}_i^{M_i*})$ maximizes $\succeq_i$ on

$$\left\{ (\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_i^{M_i*}) \in X_i \mid \tilde{\mathbf{p}}^{L*}\tilde{\mathbf{x}}_i^L + \sum_{k\in\mathcal{K}} \tilde{\tau}_i^{k*} \leq \tilde{\mathbf{p}}^{L*}\boldsymbol{\omega}_i^L + \sum_{j\in\mathcal{J}} \theta_{ij}(\tilde{\mathbf{p}}^{L*}, \tilde{\mathbf{p}}^{N*})\tilde{\mathbf{y}}_j^* \right.$$
$$\left. + \sum_{k\in\mathcal{K}} \theta_{iJ+k} \left[\sum_{i\in\mathcal{I}} \tilde{\tau}_i^{k*} + \tilde{\mathbf{p}}^{N*}\tilde{\mathbf{s}}_k^{N*} \right] \right\}.$$

(iii) For all $j \in \mathcal{J}$, $\tilde{\mathbf{y}}_j^* \in \arg\max_{\mathbf{y}_j \in Y_j} (\tilde{\mathbf{p}}^{L*}, \tilde{\mathbf{p}}^{N*})\mathbf{y}_j$.

(iv) For all $k \in \mathcal{K}$, there do not exist a $\mathbf{s}_k' \in S_k$ and $\langle \mathbf{x}_i' \rangle_{i\in\mathcal{I}}$ with $\mathbf{x}_i' \in X_i$ for all $i \in \mathcal{I}$ such that

(a) $x_{km}^{M_i\prime} = s_{km}^{M_i\prime} + \sum_{k'\neq k} \tilde{s}_{k'm}^{M_i*}$ for all $i \in \mathcal{I}$ and all $m \in \mathcal{M}$.

(b) $\mathbf{x}_i^{L\prime} = \mathbf{x}_i^{L*}$ for all $i \in \mathcal{I}$.

(c) $\mathbf{x}_i' \succeq_i \tilde{\mathbf{x}}_i^*$ for all $i \in \mathcal{I}$ and $\mathbf{x}_{i'}' \succ_{i'} \tilde{\mathbf{x}}_{i'}^*$ for some $i' \in \mathcal{I}$.

(d) $\tilde{\mathbf{p}}^{L*} \sum_{i\in\mathcal{I}} \tilde{\mathbf{x}}_i^{L\prime} \leq \tilde{\mathbf{p}}^{L*} \sum_{i\in\mathcal{I}} \boldsymbol{\omega}_i^L + (\tilde{\mathbf{p}}^L, \tilde{\mathbf{p}}^N) \sum_{j\in\mathcal{J}} \tilde{\mathbf{y}}_j^* + \tilde{\mathbf{p}}^{N*} \left[s_k^{N\prime} + \sum_{k'\neq k} \tilde{s}_{k'}^{N*} \right].$

Part (i) of Condition $2''$ is the requirement that the equilibrium allocation is feasible. Part (ii) requires each consumer's bundle of private goods to maximize utility subject to his budget constraint given the equilibrium quantities of the shared goods that he is provided with. In this constraint, the term in square brackets is group k's profits, which is its revenue from the lump-sum payments for the shared goods minus the cost of its shared good facility inputs. Part (iii), like its counterpart in a competitive shared goods equilibrium, requires firms to be competitive profit maximizers.

In Part (iv), each group is assumed to behave in a cooperative manner—not as competitive profit maximizer (as is the case in a competitive shared goods equilibrium). When group k chooses its equilibrium sharing plan $\tilde{\mathbf{s}}_k^*$, it is not possible to unilaterally choose a new sharing plan in S_k in such a way that (1) the resulting changes to the consumers' consumptions of the shared goods is a Pareto improvement and (2) using the equilibrium prices $\tilde{\mathbf{p}}^*$, the consumers taken together can afford to pay for the private goods they consume and for the cost of the shared good facility inputs.[21]

Consider a competitive shared goods equilibrium

$$(\mathbf{a}^*, \mathbf{p}^*) = (\langle \mathbf{x}_i^* \rangle_{i \in \mathcal{I}}, \langle \mathbf{y}_j^* \rangle_{j \in \mathcal{J}}, \langle \mathbf{s}_k^* \rangle_{k \in \mathcal{K}}, \mathbf{p}^*)$$

for the economy $\mathcal{D}$. Let

$$(\tilde{\mathbf{a}}^{**}, \tilde{\mathbf{p}}^{**}, \tilde{\boldsymbol{\tau}}^{**}) = (\langle \mathbf{x}_i^* \rangle_{i \in \mathcal{I}}, \langle \mathbf{y}_j^* \rangle_{j \in \mathcal{J}}, \langle \mathbf{s}_k^* \rangle_{k \in \mathcal{K}}, \langle \mathbf{p}^{i*} \rangle_{i \in \mathcal{I}}, \langle \mathbf{p}^{j*} \rangle_{j \in \mathcal{J}}, \tilde{\boldsymbol{\tau}}^*),$$

where for all $i \in \mathcal{I}$ and all $k \in \mathcal{K}$,

$$\tau_i^{k*} = \sum_{m \in \mathcal{M}} p_m^{M_i *} s_{km}^{M_i *}.$$

That is, τ_i^{k*} is the expenditure of consumer i on shared goods in the competitive shared goods equilibrium. In $(\tilde{\mathbf{a}}^{**}, \tilde{\mathbf{p}}^{**}, \tilde{\boldsymbol{\tau}}^{**})$, the allocation and the prices for the private and shared goods facilities are the same as in $(\mathbf{a}^*, \mathbf{p}^*)$ but the prices for the shared goods are replaced by the lump-sum charges $\tilde{\boldsymbol{\tau}}^*$. In Theorem 7, it is shown that if the economy $\mathcal{E}$ satisfies Assumption 1, then $(\tilde{\mathbf{a}}^{**}, \tilde{\mathbf{p}}^{**}, \tilde{\boldsymbol{\tau}}^{**})$ is a cooperative shared goods equilibrium.[22]

Theorem 7. *If the economy $\mathcal{E}$ satisfies Assumption 1 and $(\mathbf{a}^*, \mathbf{p}^*)$ is a competitive shared goods equilibrium for $\mathcal{E}$, then $(\tilde{\mathbf{a}}^{**}, \tilde{\mathbf{p}}^{**}, \tilde{\boldsymbol{\tau}}^{**})$ is a cooperative shared goods equilibrium for $\mathcal{E}$.*

[21]In his definition of a public competitive equilibrium, Foley (1967, p. 59; 1970, p. 67) also requires each consumer's private consumptions in the new proposal to be affordable for him at the equilibrium prices. See also Florenzano (2009, p. 6). This proviso can be added to Condition 2″.(iv) without affecting the results presented in this section.

[22]See Florenzano (2009, Proposition 2.1) for the corresponding result for public goods.

Proof. Because $(\mathbf{a}^*, \mathbf{p}^*)$ is a competitive shared goods equilibrium for $\mathcal{E}$, Condition 2 is satisfied. Given how $(\tilde{\mathbf{a}}^{**}, \tilde{\mathbf{p}}^{**}, \tilde{\boldsymbol{\tau}}^{**})$ is constructed from $(\mathbf{a}^*, \mathbf{p}^*)$, Conditions 2″.(i), 2″.(ii), and 2″.(iii) trivially follow from their counterparts in Condition 2 once it is noted that each group's profits and each consumer's expenditure on the shared goods provided by any group are the same in both Conditions 2.(iii) and 2″.(iii).

It remains to show that Condition 2″.(iv) is satisfied by $(\tilde{\mathbf{a}}^{**}, \tilde{\mathbf{p}}^{**}, \tilde{\boldsymbol{\tau}}^{**})$. On the contrary, suppose that there exist $\mathbf{s}'_k \in S_k$ and $\langle \mathbf{x}'_i \rangle_{i \in \mathcal{I}}$ with $\mathbf{x}'_i \in X_i$ for all $i \in \mathcal{I}$ such that the four parts of Condition 2″.(iv) hold. Because preferences are locally non-satiated, for all $i \in \mathcal{I}$, the budget constraints in Conditions 2.(ii) and 2″.(ii) hold with equality,

$$\mathbf{x}'_i \succeq_i \tilde{\mathbf{x}}_i^{**} \to \tilde{\mathbf{p}}^{L**}\tilde{\mathbf{x}}_i^{L\prime} + \tilde{\mathbf{p}}^{M_i**}\tilde{\mathbf{x}}_i^{M_i\prime} \geq \tilde{\mathbf{p}}^{L**}\tilde{\mathbf{x}}_i^{L**} + \tilde{\mathbf{p}}^{M_i**}\tilde{\mathbf{x}}^{M_i**},$$

and

$$\mathbf{x}'_i \succ_i \tilde{\mathbf{x}}_i^{**} \to \tilde{\mathbf{p}}^{L**}\tilde{\mathbf{x}}_i^{L\prime} + \tilde{\mathbf{p}}^{M_i**}\tilde{\mathbf{x}}_i^{M_i\prime} > \tilde{\mathbf{p}}^{L**}\tilde{\mathbf{x}}_i^{L**} + \tilde{\mathbf{p}}^{M_i**}\tilde{\mathbf{x}}^{M_i**}i.$$

Thus,

$$\tilde{\mathbf{p}}^{L**}\sum_{i\in\mathcal{I}}\tilde{\mathbf{x}}_i^{L\prime} + \tilde{\mathbf{p}}^{M_i**}\sum_{i\in\mathcal{I}}\tilde{\mathbf{x}}_i^{M_i\prime} > \tilde{\mathbf{p}}^{L**}\sum_{i\in\mathcal{I}}\tilde{\mathbf{x}}_i^{L**} + \sum_{i\in\mathcal{I}}\tilde{\mathbf{p}}^{M_i**}\tilde{\mathbf{x}}_i^{M_i**}$$

or, equivalently,

$$\tilde{\mathbf{p}}^{L**}\sum_{i\in\mathcal{I}}\tilde{\mathbf{x}}_i^{L\prime} > \tilde{\mathbf{p}}^{L**}\sum_{i\in\mathcal{I}}\tilde{\mathbf{x}}_i^{L**} + \tilde{\mathbf{p}}^{M**}\sum_{i\in\mathcal{I}}\tilde{\mathbf{x}}_i^{M**} - \sum_{i\in\mathcal{I}}\tilde{\mathbf{p}}^{M_i**}\tilde{\mathbf{x}}_i^{M_i\prime}.$$

Using the observation that the budget constraints hold with equality and the expressions for the groups' profits, this inequality is equivalent to

$$\tilde{\mathbf{p}}^{L**}\sum_{i\in\mathcal{I}}\tilde{\mathbf{x}}_i^{L\prime} > \tilde{\mathbf{p}}^{L**}\sum_{i\in\mathcal{I}}\boldsymbol{\omega}_i^L + \sum_{j\in\mathcal{J}}\tilde{\mathbf{p}}^{j**}\tilde{\mathbf{y}}_j^{**} + \tilde{\mathbf{p}}^{N**}\left[s_k^{N\prime} + \sum_{k'\neq k}\tilde{s}_{k'}^{N**}\right],$$

which contradictions Condition 2″.(iv).(d). □

By Theorem 6, if the economy $\mathcal{E}$ satisfies Assumptions 1–7, then a competitive shared goods equilibrium $C(\mathcal{D})$ exists for $\mathcal{E}$. Hence, the existence of a cooperative shared goods equilibrium for this economy follows trivially from Theorems 6 and 7.

Theorem 8. *If the economy $\mathcal{E}$ satisfies Assumptions 1–7, then a cooperative shared goods equilibrium $\tilde{C}(\mathcal{D})$ exists for $\mathcal{E}$.*

9. Concluding Remarks

The use of a sharing technology to model goods that embody different degrees of rivalry in consumption sheds light on the restrictive nature of the sharing possibilities afforded by pure public goods. Modeling sharing as a technological process highlights the affinity shared goods provision has to the production of private goods in the theory of joint supply pioneered by Marshall (1920), an observation that has been exploited by Buchanan (1966, 1968) and Weymark (2004) to study impure public goods. Sharing technologies also offer a new perspective on the strategy that Fabre-Sender (1969) and Foley (1970) use to establish the existence of a Lindahl equilibrium—in effect, the mapping they use to to construct an associated economy with personalized consumptions of public goods treats public provision as governed by a sharing technology for public goods. It is this observation that is generalized here to accommodate different possible sharing arrangements and that underlies the strategy used to prove the existence of a competitive shared goods equilibrium.

For both of the shared goods equilibrium concepts considered here, it is a straightforward to show that any equilibrium allocation is Pareto optimal. For the case of public goods, Florenzano (2009) shows that any Pareto optimal allocation can be decentralized as either a Lindahl equilibrium or as a public competitive equilibrium if suitable lump-sum transfers between the consumers are carried out provided that no consumer regards any public good as a bad and no firm uses any public good as an input. In the model with shared goods considered here, no shared goods are used as inputs by the firms.[23] So, it is reasonable to conjecture that any Pareto optimal allocation with shared goods can be decentralized as either a competitive shared goods equilibrium or as a cooperative shared goods competitive equilibrium when combined with suitable lump-sum transfers if, in addition to the assumptions made here, a consumer's preferences are are assumed to be monotonic is his consumption of the shared goods. It is beyond the scope of this article to examine the validity of this conjecture.

Two mechanisms have been considered for the provision of shared goods. One of them generalizes the competitive market mechanism used by Lindahl (1958) for public goods. The other is cooperative in nature and is similar in

[23]Both of these assumptions are satisfied in the decentralization theorem for pure public goods established by Foley (1967, p. 61).

spirit to public competitive equilibrium mechanism for the provision of public goods proposed by Foley (1967, 1970). In the cooperative mechanism considered here, groups make their shared goods provision decisions to ensure that that no Pareto improvements that they can implement are left unexploited. Other mechanisms for shared goods provision are possible and worth exploring. For example, voting could be used to determine how shared goods are provided. Also, the way that the provision of shared goods is financed could be modified by, for example, using taxes based on income or wealth, as is done by Foley (1967) for public goods.

In the Tiebout (1956) model of local public goods, a jurisdiction provides goods that are public within its boundaries, with non-residents excluded from their consumption. Tiebout contends that the free entry of new jurisdictions combined with costless mobility of individuals provides a market-like mechanism for the efficient provision of local public goods that avoids the problems of preference revelation and free riding associated with public goods. Informally, a *Tiebout equilibrium* uses competitive markets to equilibrate the demand and supply of private goods with the costs of a jurisdiction's local public goods shared among its residents in such a way that there is no incentive for a new jurisdiction to form that would attract residents from elsewhere. The formal analysis of Tiebout's proposal using general equilibrium theory was initiated by Wooders (1978). Her article has lead to an extensive literature that has deepened our understanding of the use of this kind of mechanism for local public good provision and of the circumstances in which Tiebout equilibria exist (at least approximately).[24] With shared goods, the analogues of jurisdictions are the groups that provide these goods. These groups are exogenously specified and, hence, do not need to concern themselves with potential entrants. As a consequence, in contrast to what is often the the case with Tiebout equilibria, it is not necessary to assume that the population is large in order to ensure that shared goods equilibria exist. Nevertheless, equilibrium existence problems would also arise with shared goods if the free entry of new groups were permitted.

Here, the incentive problems that arise because preferences are private information have been bracketed. Such issues have received considerable

[24]See Chan and van den Nouweland (2025) for a recent contribution to this literature that also provides an extensive discussion of earlier research on this issue. Some of these contributions use personalized prices for the local public goods. For example, to help characterize Tiebout equilibria in her model, Wooders (1989) introduces an auxiliary competitive equilibrium concept in which agents pay Lindahl prices for these goods.

attention in the mechanism design literature concerned with the provision of public goods (e.g., Green and Laffont, 1979; Florenzano, 2010). Whether the lessons learned from the study of incentive issues that arise in the provision of public goods because of asymmetries in information have much applicability to shared goods provision is an open question.

References

Arrow, K. J. (1969). The organization of economic activity: Issues pertinent to the choice of market versus non-market allocation. In *The Analysis and Evaluation of Public Expenditures: The PPB System*, Volume 1, pp. 47–64. Washington: Joint Economic Committee, Congress of the United States.

Bergstrom, T. C. (1976). Collective choice and the Lindahl allocation method. In S. A. Y. Lin (Ed.), *Theory and Measurement of Economic Externalities*, pp. 111–131. New York: Academic Press.

Bonnisseau, J.-M., E. L. del Mercato, and P. Siconolfi (2023). Existence of an equilibrium in Arrowian markets for consumption externalities. *Journal of Economic Theory 209*. Article no. 105638.

Buchanan, J. M. (1965). An economic theory of clubs. *Economica 32*, 1–14.

Buchanan, J. M. (1966). Joint supply, externality and optimality. *Economica 33*, 405–414.

Buchanan, J. M. (1968). *The Demand and Supply of Public Goods*. Chicago: Rand McNally.

Chan, N. W. and A. van den Nouweland (2025). Local public good equilibrium. *Games and Economic Behavior 149*, 32–42.

Debreu, G. (1959). *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. New York: Wiley.

Debreu, G. (1962). New concepts and techniques for equilibrium analysis. *International Economic Review 3*, 257–273.

Drèze, J. H. (1974). Investment under private ownership: Optimality, equilibrium and stability. In J. H. Drèze (Ed.), *Allocation Under Uncertainty: Equilibrium and Optimality*, pp. 129–166. London: Macmillan.

Fabre-Sender, F. (1969). Biens collectifs et biens à qualité variable. Unpublished manuscript, CEPREMAP.

Florenzano, M. (2009). Walras–Lindahl–Wicksell: What equilibrium concept for public goods provision? The convex case. HAL ID halshs-0053143, Hyper articles en ligne, Le Centre pour la Communication Scientifique Directe.

Florenzano, M. (2010). Government and the provision of public goods: From equilibrium models to mechanism design. *European Journal of the History of Economic Thought 17*, 1047–1077.

Foley, D. (1967). Resource allocation and the public sector. *Yale Economic Essays 7*, 42–98.

Foley, D. (1970). Lindahl's solution and the core of an economy with public goods. *Econometrica 38*, 66–72.

Green, J. R. and J.-J. Laffont (1979). *Incentives in Public Decsion-Making.* Amsterdam: North-Holland.

Hart, O. D. and H. W. Kuhn (1975). A proof of the existence of equilibrium without the free disposal assumption. *Journal of Mathematical Economics 2*, 335–342.

Lindahl, E. (1958). Just taxation — a positive solution. In R. A. Musgrave and A. T. Peacock (Eds.), *Classics in the Theory of Public Finance*, pp. 168–177. London: Macmillan. Translated by Elizabeth Henderson. Originally published in 1919 as "Positive Lösung" in *Die Gerechtigkeit der Besteuerung: eine Analysis der Steuerprinzipien auf Grundlage der Grenznutzentheorie*, Part I, Chap. 4, pp. 85–98. Lund: Gleerupska Universitets-Bokhandeln.

Marshall, A. (1920). *Principles of Economics* (Eighth ed.). London: Macmillan. First edition published in 1890.

Milleron, J.-C. (1972). Theory of value with public goods: A survey article. *Journal of Economic Theory 5*, 419–477.

Roberts, D. J. (1974). The Lindahl solution for economies with public goods. *Journal of Public Economics 3*, 23–42.

Tiebout, C. M. (1956). A pure theory of local expenditures. *J Political Econ 64*, 416–4244.

van den Nouweland, A. (2015). Lindahl and equilibrium. In C. Binder, G. Codognato, M. Teschl, and Y. Xu (Eds.), *Individual and Collective Choice and Social Welfare: Essays in Honor of Nick Baigent*, pp. 335–362. Heidelberg: Springer.

van den Nouweland, A., S. Tijs, and M. H. Wooders (2002). Axiomatization of ratio equilibria in public good economies. *Social Choice and Welfare 19*, 627–636.

Walras, L. (1954). *Elements of Pure Economics or the Theory of Social Wealth*. London: George Allen and Unwin. Translated by William Jaffé from the 1926 fourth edition of *Eléments d'économie politique pure ou théorie de richesse sociale*. Paris: R. Pichon et R. Durand-Auzias and

Lausanne: R. Rouge, Libraire-Éditeur. First edition published in 1874.

Weymark, J. A. (2004). Shared goods: A technological analysis. *Annales d'Éonomie et Statistique 75/76*, 175–195.

Wooders, M. (1978). Equilibria, the core, and jurisdictional structures in economies with a local public good. *Journal of Economic Theory 18*, 328–348.

Wooders, M. H. (1989). A Tiebout theorem. *Mathematical Social Sciences 18*, 33–55.